

# A Folkman Linear Family\*

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## Abstract

For graphs  $F$  and  $G$ , let  $F \rightarrow (G, G)$  signify that any red/blue edge coloring of  $F$  contains a monochromatic  $G$ . Define Folkman number  $f(G; p)$  to be the smallest order of a graph  $F$  such that  $F \rightarrow (G, G)$  and  $\omega(F) \leq p$ . It is shown that  $f(G; p) \leq cn$  for graphs  $G$  of order  $n$  with  $\Delta(G) \leq \Delta$ , where  $\Delta \geq 3$ ,  $c = c(\Delta)$  and  $p = p(\Delta)$  are positive constants.

**Keywords:** Folkman number; Folkman linear; Multi-partite regularity lemma

## 1 Introduction

For graphs  $F$  and  $G$ , let  $F \rightarrow (G, G)$  signify that any red/blue edge coloring of  $F$  contains a monochromatic  $G$ . The Ramsey number  $R(G)$  is the smallest  $N$  such that  $K_N \rightarrow (G, G)$ . For most graphs  $G$ , it is difficult to determine the behavior of  $R(G)$ , and even more difficult if the edge-colored graphs are restricted within that of smaller cliques instead of the complete graphs.

Define a family  $\mathcal{F}(G; p)$  of graphs as

$$\mathcal{F}(G; p) = \{F : F \rightarrow (G, G) \text{ and } \omega(F) \leq p\},$$

where  $\omega(G)$  is the clique number of  $G$ , and define

$$f(G; p) = \min\{|V(F)| : F \in \mathcal{F}(G; p)\},$$

which is called the Folkman number. We admit that  $f(G; p) = \infty$  if  $\mathcal{F}(G; p) = \emptyset$ , and thus  $f(G; p) = \infty$  if  $p < \omega(G)$ .

The investigation was motivated by a question of Erdős and Hajnal [9] who asked what was the minimum  $p$  such that  $\mathcal{F}(K_3; p) \neq \emptyset$ . An important result of Folkman [10] states that  $\mathcal{F}(K_n; p) \neq \emptyset$  for  $p \geq n$ , which was generalized by Nešetřil and Rödl [20] as  $\mathcal{F}(G; p) \neq \emptyset$  for  $p \geq \omega(G)$ . The following property is clear.

**Lemma 1** *The function  $f(G; p)$  is decreasing on  $p$ , and if  $p \geq R(G)$ , then  $f(G; p) = R(G)$ .*

Graham [14] proved that  $f(K_3; 5) = 8$  by showing  $K_8 \setminus C_5 \not\rightarrow (K_3, K_3)$ . Irving [15] proved that  $f(K_3; 4) \leq 18$ , and it was further improved by Khadzhiivanov and Nenov [16] to  $f(K_3; 4) \leq 16$ . Finally,

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Piwakowski, Radziszowski, and Urbanski [13] and Lin [18] proved  $f(K_3; 4) = 15$ . However, both upper bounds of Folkman and of Nešetřil and Rödl for  $f(K_3; 3)$  are extremely large. Frankl and Rödl [11] first gave a reasonable bound  $f(K_3; 3) \leq 7 \times 10^{11}$ . Erdős set a prize of \$100 for the challenge  $f(K_3; 3) \leq 10^{10}$ . This reward was claimed by Spencer [10, 11], who proved that  $f(K_3; 3) < 3 \times 10^9$ . Erdős then offered another \$100 prize (see [3], page 46) for the new challenge  $f(K_3; 3) < 10^6$ . Chung and Graham [3] conjectured further  $f(K_3; 3) < 10000$ , which was confirmed by Lu [19] with  $f(K_3; 3) < 9697$ , and by Dudek and Rödl [7] with more computer aid.

Let us call a family  $\mathcal{G}$  of graphs  $G_n$  of order  $n$  to be Ramsey linear if there exists a constant  $c = c(\mathcal{G}) > 0$  such that  $R(G_n) \leq cn$  for any  $G_n \in \mathcal{G}$ . Similarly, we call  $\mathcal{G}$  to be Folkman  $p$ -linear if  $f(G_n; p) \leq cn$  for any  $G_n \in \mathcal{G}$ , where  $p$  is a constant. Let  $\Delta(G_n)$  be the maximum degree of  $G_n$  of order  $n$  and set a family of graphs as

$$\mathcal{G}_\Delta = \{G_n \mid \Delta(G_n) \leq \Delta\}.$$

A result of Chvátal, Rödl, Szemerédi and Trotter [5] is as follows.

**Theorem 1** *The family  $\mathcal{G}_\Delta$  is Ramsey linear.*

The proof of Theorem 1 is a remarkable application of Szemerédi regularity lemma, in which they used the general form of the lemma. In order to generalize Theorem 1 to Folkman number, we shall have a multi-partite regularity lemma as follows.

**Theorem 2** *For any  $\epsilon > 0$  and integers  $m \geq 1$  and  $p \geq 2$ , there exists an  $M = M(\epsilon, m, p)$  such that each  $p$ -partite graph  $G(V^{(1)}, \dots, V^{(p)})$  with  $|V^{(s)}| \geq M$ ,  $1 \leq s \leq p$ , has a partition  $\{V_1^{(s)}, \dots, V_k^{(s)}\}$  for each  $V^{(s)}$ , where  $k$  is same for each part  $V^{(s)}$  and  $m \leq k \leq M$ , such that*

- (1)  $||V_i^{(s)}| - |V_j^{(s)}|| \leq 1$  for each  $s$ ;
- (2) All but at most  $\epsilon k^2 \binom{p}{2}$  pairs  $(V_i^{(s)}, V_j^{(t)})$ ,  $1 \leq s < t \leq p$ ,  $1 \leq i, j \leq k$ , are  $\epsilon$ -regular.

Using the above Theorem 2, we can deduce the following result on the Folkman  $p$ -linearity of  $\mathcal{G}_\Delta$  for some fixed  $p$ .

**Theorem 3** *Let  $\Delta \geq 3$  be an integer and  $p = R(K_\Delta)$ . Then the family  $\mathcal{G}_\Delta$  is Folkman  $p$ -linear.*

Note that for sub-family  $\mathcal{G}_{\Delta, \chi}$  consisting of  $G \in \mathcal{G}_\Delta$  with  $\chi(G) \leq \chi$ , we can take  $p = R(K_\chi)$  such that  $\mathcal{G}_{\Delta, \chi}$  is Folkman  $p$ -linear. A natural problem is asking what is a smaller  $p$  such that  $\mathcal{G}_\Delta$  is Folkman  $p$ -linear.

For an integer  $r \geq 2$ , we call an edge coloring of a graph by  $r$  colors as an  $r$ -edge coloring of the graph. For graphs  $F$  and  $G$ , let  $F \rightarrow (G)_r$  signify that any  $r$ -edge coloring of  $F$  contains a monochromatic  $G$ . Thus  $R_r(G)$  is the smallest  $N$  such that  $K_N \rightarrow (G)_r$ , and  $f_r(G; p)$  is the smallest  $N$  such that there exists a graph  $F$  of order  $N$  with  $\omega(F) = p$  satisfying  $F \rightarrow (G)_r$ . Theorem 3 can be generalized as follows.

**Theorem 4** *Let  $\Delta \geq 3$  and  $r \geq 2$  be integers and  $p = R_r(K_\Delta)$ . Then, there is some constant  $c = c(\Delta, r) > 0$  such that  $f_r(G_n, p) \leq cn$  for any  $G_n \in \mathcal{G}_\Delta$ .*

## 2 Multi-partite regularity lemma

Let  $A$  be a set of positive integers and  $A_n = A \cap \{1, \dots, n\}$ . In the 1930s, Erdős and Turán conjectured that if  $\overline{\lim}_{n \rightarrow \infty} \frac{|A_n|}{n} > 0$ , then  $A$  contains arbitrarily long arithmetic progressions. The conjecture in case of length 3 was proved by Roth [22, 23]. The full conjecture was proved by Szemerédi [26] with a deep and complicated combinatorial argument. In the proof he used a result, which is now called the bipartite regularity lemma, and then he proved the general regularity lemma in [27]. The lemma has become a totally new tool in extremal graph theory. Sometimes the regularity lemma is called uniformity lemma,

see e.g., Bollobás [2] and Gowers [13]. For many applications, we refer the readers to the survey of Komlós and Simonovits [17]. In this note, we shall discuss multi-partite regularity lemma in slightly different forms.

Let  $G(U, V)$  be a bipartite graph on two color classes  $U$  and  $V$ . For  $X \subseteq U$  and  $Y \subseteq V$ , denote by  $e(X, Y)$  the number of edges between  $X$  and  $Y$  of  $G$ . The ratio

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}$$

is called the edge density of  $(X, Y)$ , which is the probability that any pair  $(x, y)$  selected randomly from  $X \times Y$  is an edge. Clearly  $0 \leq d(X, Y) \leq 1$ .

The first form of regularity lemma given by Szemerédi in [26] is as follows, in which corresponding to each subset  $U_i$  in the partition of  $U$ , we have to choose its own partition  $V_{i,j}$  of  $V$ .

**Lemma 2 (Bipartite Regularity Lemma-Old Form)** *For any positive  $\epsilon_1, \epsilon_2, \delta, \rho_1, \rho_2$ , there exist  $k_1, k_2, M_1, M_2$  such that every bipartite graph  $G(U, V)$  with  $|U| > M_1$  and  $|V| > M_2$ , there exist disjoint  $U_i \subset U$ ,  $i < k_1$ , and for each  $i < k_1$ , disjoint  $V_{i,j} \subset V$ ,  $j < k_2$ , such that:*

- (1)  $|U - \cup_{i < k_1} U_i| < \rho_1|U|$ , and  $|V - \cup_{j < k_2} V_{i,j}| < \rho_2|V|$  for any  $i < k_1$ ;
- (2) For all  $i < k_1$ ,  $j < k_2$ ,  $X \subseteq U_i$  and  $Y \subseteq V_{i,j}$  with  $|X| > \epsilon_1|U_i|$  and  $|Y| > \epsilon_2|V_{i,j}|$ , we have

$$d(X, Y) \geq d(U_i, V_{i,j}) - \delta;$$

- (3) For all  $i < k_1$ ,  $j < k_2$  and  $x \in U_i$ ,  $|N(x) \cap V_{i,j}| \leq (d(U_i, V_{i,j}) + \delta)|V_{i,j}|$ .

For  $\epsilon > 0$ , a disjoint pair  $(X, Y)$  is called  $\epsilon$ -regular if any  $X' \subseteq X$  and  $Y' \subseteq Y$  with  $|X'| > \epsilon|X|$  and  $|Y'| > \epsilon|Y|$  satisfy

$$|d(X, Y) - d(X', Y')| \leq \epsilon.$$

We shall call  $U_0 = U - \cup_{i < k_1} U_i$ , and  $V_0 = V - \cup_{j < k_2} V_{i,j}$  in Theorem 2 to be the *exceptional sets*. The following is the general regularity lemma of Szemerédi [27], in which the partition  $C_0, C_1, \dots, C_k$  is *equitable* in sense of that all sets  $C_i$  other than the exceptional set  $C_0$  have the same size.

**Lemma 3 (General Regularity Lemma)** *For any  $\epsilon > 0$  and any  $m \geq 1$ , there exists  $M = M(\epsilon, m) > m$  such that every graph  $G$  of order at least  $m$  has a partition  $C_0, C_1, \dots, C_k$  with  $m \leq k \leq M$  such that*

- (1)  $|C_1| = |C_2| = \dots = |C_k|$  and  $|C_0| \leq \epsilon n$ ;
- (2) All but at most  $\epsilon k^2$  pairs  $(C_i, C_j)$  with  $1 \leq i < j \leq k$  are  $\epsilon$ -regular.

There are many generalizations of Szemerédi regularity lemma, in particular, Frankl and Rödl [12] generalized it to hypergraphs and later Chung [4] formulated regularity lemma on  $t$ -uniform hypergraphs when discussing the problems of quasi-random hypergraphs.

The regularity lemma has numerous applications in various areas, mainly in extremal graph theory such as [5] by Chvátal, Rödl, Szemerédi and Trotter. In an application, Eaton and Rödl [8] obtained a form of the regularity lemma for  $p$ -partite  $p$ -uniform hypergraph. To state their result for multi-partite graph, let us have some definitions.

Let  $G(V^{(1)}, \dots, V^{(p)})$  be a  $p$ -partite graph on vertex set  $\cup_{i=1}^p V^{(i)}$ . Consider partitions of the set  $V^{(1)} \times \dots \times V^{(p)}$ , where each partition class is of the form  $W_1 \times \dots \times W_p$ ,  $W_i \subseteq V^{(i)}$ ,  $1 \leq i \leq p$ , which is called *cylinders*. Let us say that a cylinder  $W_1 \times \dots \times W_p$  is  $\epsilon$ -regular if the subgraph of  $G$  induced on the set  $\cup_{i=1}^p W_i$  is such that all pairs  $(W_i, W_j)$ ,  $1 \leq i < j \leq p$ , are  $\epsilon$ -regular.

Eaton and Rödl stated their result with exceptional  $p$ -tuples instead of exceptional sets, for which Alon, Duke, Lefmann, Rödl and Yusterk studied the computational difficulty of finding such a regular partition in [1].

**Lemma 4** Let  $G(V^{(1)}, \dots, V^{(p)})$  be a  $p$ -partite graph with  $|V^{(i)}| = n$ ,  $i = 1, \dots, p$ . Then for every  $\epsilon > 0$  there exists a partition of  $V^{(1)} \times \dots \times V^{(p)}$  into  $k$  cylinders with  $k \leq 4^h$ , where  $h = \frac{\binom{p}{2}}{\epsilon^8}$ , such that all but at most  $\epsilon n^p$  of the  $p$ -tuples  $(v_1, \dots, v_p)$  of  $V^{(1)} \times \dots \times V^{(p)}$  are in  $\epsilon$ -regular cylinders of the partition.

Note that in Lemma 4, the transverse section  $\{W_i\}$  of the partition is a partition of  $V^{(i)}$ , which may be not equitable, and for  $i \neq j$ , the numbers of subsets in the partitions  $\{W_i\}$  and  $\{W_j\}$  may be different. We shall have a multi-partite regularity lemma as follows.

**Lemma 5** For any  $\epsilon > 0$  and integers  $m \geq 1$  and  $p \geq 2$ , there exists  $M = M(\epsilon, m, p)$  such that each  $p$ -partite graph  $G(V^{(1)}, \dots, V^{(p)})$  with  $|V^{(s)}| \geq M$ ,  $1 \leq s \leq p$ , has a partition  $\{V_0^{(s)}, V_1^{(s)}, \dots, V_k^{(s)}\}$  for each  $V^{(s)}$ , where  $k$  is same for each part  $V^{(s)}$  and  $m \leq k \leq M$ , such that

- (1)  $|V_1^{(s)}| = \dots = |V_k^{(s)}|$  and  $|V_0^{(s)}| \leq \epsilon |V^{(s)}|$  for each  $s$ ;
- (2) All but at most  $\epsilon k^2 \binom{p}{2}$  pairs  $(V_i^{(s)}, V_j^{(t)})$ ,  $1 \leq s < t \leq p$ ,  $1 \leq i, j \leq k$ , are  $\epsilon$ -regular.

The following multicolor multi-partite regularity lemma is an analogy of Theorem 2, which is needed for proof of Theorem 4.

**Lemma 6** For any  $\epsilon > 0$  and integers  $m \geq 1$ ,  $p \geq 2$  and  $r \geq 1$ , there exists an  $M = M(\epsilon, m, p, r)$  such that if the edges of a  $p$ -partite graph  $G(V^{(1)}, \dots, V^{(p)})$  with  $|V^{(s)}| \geq M$ ,  $1 \leq s \leq p$  are  $r$ -colored, then all monochromatic graphs have the same partition  $\{V_1^{(s)}, \dots, V_k^{(s)}\}$  for each  $V^{(s)}$ , where  $k$  is same for each part  $V^{(s)}$  and  $m \leq k \leq M$ , such that

- (1)  $||V_i^{(s)}| - |V_j^{(s)}|| \leq 1$  for each  $s$ ;
- (2) All but at most  $\epsilon k^2 r \binom{p}{2}$  pairs  $(V_i^{(s)}, V_j^{(t)})$ ,  $1 \leq s < t \leq p$ ,  $1 \leq i, j \leq k$ , are  $\epsilon$ -regular in each monochromatic graph.

### 3 Proofs for multi-partite regularity lemma

In this section, we prove Lemma 5, Theorem 2 and Lemma 6. To reduce the complicity of notations in the proofs, we shall prove them in case  $p = 2$ , which are bipartite regularity lemmas.

**Lemma 7** Let  $G(U, V)$  be a bipartite graph and let  $X \subseteq U$  and  $Y \subseteq V$ . If  $X' \subseteq X$  and  $Y' \subseteq Y$  satisfy  $|X'| > (1 - \delta)|X|$  and  $|Y'| > (1 - \delta)|Y|$ , then

$$|d(X', Y') - d(X, Y)| < 2\delta \quad \text{and} \quad |d^2(X', Y') - d^2(X, Y)| < 4\delta.$$

A crucial point for the regularity lemma is that the number  $k$  of classes in partition is bounded for any graph. For proofs, we need the well-known defect form of Cauchy-Schwarz inequality.

**Lemma 8** Let  $d_i$  be reals and  $s > t \geq 1$  be integers. If

$$\frac{1}{s} \sum_{i=1}^s d_i = \frac{1}{t} \sum_{i=1}^t d_i + \delta,$$

then

$$\frac{1}{s} \sum_{i=1}^s d_i^2 \geq \left( \frac{1}{s} \sum_{i=1}^s d_i \right)^2 + \frac{t\delta^2}{s-t}.$$

Let  $G(U, V)$  be a bipartite graph, a partition

$$\mathcal{P} = \left\{ U_i, V_j \mid 0 \leq i, j \leq k \right\},$$

where  $U = \cup_{p=1}^k U_i$  and  $V = \cup_{p=1}^k V_i$ , is called to be an *equitable* partition of  $U \cup V$  with exceptional classes  $U_0$  and  $V_0$  if  $|U_i| = |U_j|$  and  $|V_i| = |V_j|$  for  $1 \leq i, j \leq k$ . For convenience, we say an equitable partition  $\mathcal{P}$  is  $\epsilon$ -regular if all but at most  $\epsilon k^2$  pairs of  $(U_i, V_j)$  are  $\epsilon$ -regular. Define

$$q(\mathcal{P}) = \frac{1}{k^2} \sum_{1 \leq i, j \leq k} d^2(U_i, V_j).$$

It is easy to see that  $0 \leq q(\mathcal{P}) \leq 1$  since  $0 \leq d(U_i, V_j) \leq 1$ .

In the following, we will show that if  $\mathcal{P}$  is not  $\epsilon$ -regular, then there is a partition  $\mathcal{P}'$  with the new exceptional classes a bit larger than the old one, but  $q(\mathcal{P}') \geq q(\mathcal{P}) + \frac{\epsilon^5}{4}$ . Do this again if  $\mathcal{P}'$  is not  $\epsilon$ -regular yet. The number of iterations is thus at most  $4/\epsilon^5$  in order to obtain an  $\epsilon$ -regular partition. Without loss of generality, we assume that  $0 < \epsilon \leq 1/2$  since if  $\epsilon > 1/2$ , one can take  $M(\epsilon, m)$  to be  $M(1/2, m)$ .

**Lemma 9** *Let  $G(U, V)$  be a bipartite graph with  $|U| = n_1 \geq M$  and  $|V| = n_2 \geq M$ , which has an equitable partition*

$$\mathcal{P} = \left\{ U_i, V_j \mid 0 \leq i, j \leq k \right\}$$

*with exceptional classes  $U_0$  and  $V_0$ . Suppose  $2^k \geq 16/\epsilon^5$ ,  $|U_i| = c_1 \geq 2^{3k}$  and  $|V_j| = c_2 \geq 2^{3k}$ . We have if  $\mathcal{P}$  is not  $\epsilon$ -regular, then there is an equitable partition*

$$\mathcal{P}' = \left\{ U'_i, V'_j \mid 0 \leq i, j \leq \ell \right\}$$

*with exceptional class  $U'_0 \supseteq U_0$  and  $V'_0 \supseteq V_0$ , and  $\ell = k(4^k - 2^k)$  satisfying*

- (1)  $|U'_0| \leq |U_0| + n_1/2^{k-1}$  and  $|V'_0| \leq |V_0| + n_2/2^{k-1}$ ;
- (2)  $q(\mathcal{P}') \geq q(\mathcal{P}) + \epsilon^5/4$ .

**Proof.** Separate all pairs  $(i, j)$ ,  $1 \leq i, j \leq k$ , of indices into  $S$  and  $T$ , corresponding with that the pair  $(U_i, V_j)$  is  $\epsilon$ -regular or not, respectively. For  $(i, j) \in S$ , set  $U_{ij} = V_{ji} = \emptyset$ , and for  $(i, j) \in T$ , set  $U_{ij} \subseteq U_i$  and  $V_{ji} \subseteq V_j$  with  $|U_{ij}| > \epsilon c_1$ ,  $|V_{ji}| > \epsilon c_2$ , and

$$|d(U_{ij}, V_{ji}) - d(U_i, V_j)| > \epsilon.$$

For fixed  $i$ ,  $1 \leq i \leq k$ , consider an equivalence relation  $\equiv$  on  $U_i$  as  $x \equiv y$  if and only if both  $x$  and  $y$  belong to the same  $U_{ij}$ 's. The equivalence classes are atoms of algebra induced by  $U_{ij}$ , and each  $U_i$  has at most  $2^k$  atoms. Similarly, each  $V_j$  has at most  $2^k$  atoms.

For  $p = 1, 2$ , set  $d_p = \lfloor c_p/4^k \rfloor$ . Let us cut each atom in  $U_i$  into pairwise disjoint  $d_1$ -subsets. Denote by  $z$  for the maximal number of these  $d_1$ -subsets that one can take, clearly  $z \geq 4^k - 2^k$  as  $z d_1 + 2^k(d_1 - 1) \geq c_1$ . Set

$$H = 4^k - 2^k,$$

and take *exactly*  $H$  such  $d_1$ -subsets and add the remainder to the “rubbish bin” to get a new exceptional set  $U'_0$ . Label all these  $d_1$ -subsets in  $U_i$  as  $D_{i1}, \dots, D_{iH}$ . Set  $U'_0 = U_0 \cup \left[ \cup_{i=1}^k \left( U_i \setminus \cup_{h=1}^H D_{ih} \right) \right]$ , and so  $|U'_0| = |U_0| + k(c_1 - H d_1)$ . Since

$$H d_1 \geq (4^k - 2^k) \left( \frac{c_1}{4^k} - 1 \right) > c_1 - \frac{c_1}{2^{k-1}}$$

by noting  $c_1 \geq 2^{3k}$ , we have  $|U'_0| \leq |U_0| + n_1/2^{k-1}$ . Rename  $D_{ih}$  as  $U'_s$  for  $1 \leq s \leq \ell$ , where  $\ell = kH$ .

Similarly, we can cut each atom in  $V_j$  into pairwise disjoint  $d_2$ -subsets and take  $H$  such subsets  $E_{j1}, \dots, E_{jH}$  in  $V_j$ . Set  $V'_0 = V_0 \cup \left[ \cup_{i=1}^k \left( V_j \setminus \cup_{h=1}^H E_{jh} \right) \right]$ , and similarly  $|V'_0| \leq |V_0| + n_2/2^{k-1}$ . Rename  $E_{jh}$  as  $V'_t$  for  $1 \leq t \leq \ell$ .

Denote the new equitable partition by

$$\mathcal{P}' = \left\{ U'_i, V'_j \mid 0 \leq i, j \leq \ell \right\}$$

of  $U \cup V$  with exceptional classes  $U'_0 \supseteq U_0$  and  $V'_0 \supseteq V_0$ . All that remains is to show  $q(\mathcal{P}') \geq q(\mathcal{P}) + \epsilon^5/4$ .

For  $1 \leq i, j \leq k$ , set

$$\begin{aligned} \overline{U}_i &= \cup_{h=1}^H D_{ih}, & \overline{U}_{ij} &= \cup \{ D_{ih} : D_{ih} \subseteq U_{ij} \}, & \text{and} \\ \overline{V}_j &= \cup_{h=1}^H E_{jh}, & \overline{V}_{ji} &= \cup \{ E_{jh} : D_{jh} \subseteq V_{ji} \}. \end{aligned}$$

Set partition  $\overline{\mathcal{P}} = \{U'_0, \overline{U}_1, \dots, \overline{U}_k; V'_0, \overline{V}_1, \dots, \overline{V}_k\}$  with exceptional class  $U'_0$  and  $V'_0$ .

*Claim 1.*  $q(\overline{\mathcal{P}}) \geq q(\mathcal{P}) - \epsilon^5/2$ .

*Proof of Claim 1.* Note that  $\frac{|U_i \setminus \overline{U}_i|}{|U_i|} < \frac{1}{2^{k-1}} < \frac{\epsilon^5}{8}$  and  $\frac{|V_j \setminus \overline{V}_j|}{|V_j|} < \frac{\epsilon^5}{8}$  for any pair  $(U_i, V_j)$ , we have

$$|d(\overline{U}_i, \overline{V}_j) - d(U_i, V_j)| \leq \frac{\epsilon^5}{4} \quad (1)$$

by Lemma 7. Hence  $d^2(\overline{U}_i, \overline{V}_j) \geq d^2(U_i, V_j) - \epsilon^5/2$ , which implies that  $q(\overline{\mathcal{P}}) \geq q(\mathcal{P}) - \epsilon^5/2$  as claimed.

*Claim 2.* If  $(i, j) \in T$ , then  $|d(\overline{U}_{ij}, \overline{V}_{ji}) - d(\overline{U}_i, \overline{V}_j)| > \frac{15}{16}\epsilon$ .

*Proof of Claim 2.* Clearly,  $\frac{|U_{ij} \setminus \overline{U}_{ij}|}{|U_{ij}|} \leq \frac{|U_i \setminus \overline{U}_i|}{|U_i|} \frac{|U_i|}{|U_{ij}|} \leq \frac{\epsilon^4}{8}$  and  $\frac{|V_{ji} \setminus \overline{V}_{ji}|}{|V_{ji}|} \leq \frac{\epsilon^4}{8}$ , which and Lemma 7 give

$$|d(\overline{U}_{ij}, \overline{V}_{ji}) - d(U_{ij}, V_{ji})| \leq \frac{\epsilon^4}{4}. \quad (2)$$

Therefore, if  $(i, j) \in T$ , the bounds (1) and (2) with the fact that  $0 < \epsilon \leq 1/2$  will yield the desired inequality.

Let us return to the partition  $\mathcal{P}'$  in which each class is either a  $d_1$ -subset  $D_{iu}$  or a  $d_2$ -subset  $E_{jv}$  except  $U'_0$  and  $V'_0$ . For any pair  $(U_i, V_j)$ ,

$$d(\overline{U}_i, \overline{V}_j) = \frac{1}{H^2} \sum_{1 \leq u, v \leq H} d(D_{iu}, E_{jv})$$

since  $|\overline{U}_i| = Hd_1$  and  $|\overline{V}_j| = Hd_2$ . Set

$$A(i, j) = \frac{1}{H^2} \sum_{1 \leq u, v \leq H} d^2(D_{iu}, E_{jv}).$$

Then from Cauchy-Schwarz inequality, for any pair  $(i, j)$ , we have

$$A(i, j) \geq d^2(\overline{U}_i, \overline{V}_j). \quad (3)$$

If  $(i, j) \in T$ , we have some gain. Let  $R = R(i, j)$  be the set of indices  $(u, v)$  such that  $D_{iu} \in \overline{U}_{ij}$  and  $E_{jv} \in \overline{V}_{ji}$ . Then

$$d(\overline{U}_{ij}, \overline{V}_{ji}) = \frac{1}{|R|} \sum_{(u, v) \in R} d(D_{iu}, E_{jv}).$$

Note that  $\frac{|R|}{H^2} = \frac{\overline{U_{ij}} \overline{V_{ji}}}{\overline{U_i} \overline{V_j}} \geq ((1 - 2^{-7})\epsilon)^2$ , So Lemma 8 and Claim 2 imply

$$A(i, j) \geq d^2(\overline{U_i}, \overline{V_j}) + \frac{|R|}{H^2} (d(\overline{U_i}, \overline{V_j}) - d(\overline{U_{ij}}, \overline{V_{ji}}))^2 \geq d^2(\overline{U_i}, \overline{V_j}) + \frac{3}{4}\epsilon^4. \quad (4)$$

Noticing that  $\ell = kH$ , we have

$$q(\mathcal{P}') = \frac{1}{\ell^2} \sum_{1 \leq s, t \leq \ell} d^2(U'_s, V'_t) = \frac{1}{k^2} \frac{1}{H^2} \sum_{1 \leq i, j \leq k} \sum_{1 \leq u, v \leq H} d^2(D_{iu}, E_{jv}) = \frac{1}{k^2} \sum_{1 \leq i, j \leq k} A(i, j).$$

Now, combine inequalities (3) and (4), and recall Claim 1 and that  $\mathcal{P}$  is not  $\epsilon$ -regular, we have

$$q(\mathcal{P}') \geq \frac{1}{k^2} \left[ \sum_{(i, j) \in S} d^2(\overline{U_i}, \overline{V_j}) + \sum_{(i, j) \in T} \left( d^2(\overline{U_i}, \overline{V_j}) + \frac{3}{4}\epsilon^4 \right) \right] \geq q(\mathcal{P}) + \frac{\epsilon^5}{4}.$$

This completes the proof of Lemma 9.  $\square$

**Proof of Lemma 5.** Let  $k_0$  be an integer such that  $k_0 \geq m$  and  $2^{-k_0} \leq \epsilon^5/16$ , and define  $k_{i+1} = k_i(4^{k_i} - 2^{k_i})$ . Set  $M_i = k_i 2^{3k_i}$  and  $M = M_t$ . Lemma 9 implies that at most  $t = 4\lfloor \epsilon^{-5} \rfloor$  iterations will yield a required partition, which completes the proof of Lemma 5.  $\square$

**Proof of Theorem 2.** For given  $\epsilon > 0$  and  $m \geq 1$ , Theorem 2 implies that there is an  $M > m$  and an equitable and  $\frac{\epsilon^2}{4}$ -regular partition  $\mathcal{P} = \{U_i, V_j | 0 \leq i, j \leq k\}$  with  $m \leq k \leq M$ . Since  $|U_0| < \frac{\epsilon^2}{4}n_1$ , we have  $\lfloor (1 - \epsilon^2/4)n_1/k \rfloor \leq |U_i| \leq n_1/k$ . Partition  $U_0$  into  $k$  classes  $U_{01}, U_{02}, \dots, U_{0k}$  such that  $|U_{0i}| = \lfloor |U_0|/k \rfloor$  or  $|U_{0i}| = \lceil |U_0|/k \rceil$ . Set  $U'_i = U_i \cup U_{0i}$ , clearly  $|U'_i| = \lfloor n_1/k \rfloor$  or  $|U'_i| = \lceil n_1/k \rceil$ . Similarly, let us partition  $V_0$  into  $k$  classes  $V_{01}, V_{02}, \dots, V_{0k}$  such that  $|V_{0i}| = \lfloor |V_0|/k \rfloor$  or  $|V_{0i}| = \lceil |V_0|/k \rceil$ . Set  $V'_i = V_i \cup V_{0i}$ , we have the sizes of any  $V'_i$  and  $V'_j$  differ at most by one. Then the Partition  $\mathcal{P}' = \{U'_i, V'_j | 0 \leq i, j \leq k\}$  is as desired by noting that if a pair  $(U_i, V_j)$  is  $\frac{\epsilon^2}{4}$ -regular, then  $(U'_i, V'_j)$  is  $\epsilon$ -regular.  $\square$

**Proof of Lemma 6.** A similar proof as Theorem 2, but modify the definition of index by summing the indices for each color,

$$q(\mathcal{P}) = \frac{1}{k^2} \sum_{1 \leq h \leq r} \sum_{1 \leq s < t \leq p} \sum_{1 \leq i, j \leq k} d^2(V_i^{(s)}, V_j^{(t)}).$$

Then we have analogy of Lemma 5 for multi-color case. Furthermore, we have Lemma 6.  $\square$

## 4 A Folkman linear family

In this section, we shall apply multi-partite regularity lemma to the Folkman numbers involving the family  $\mathcal{G}_\Delta$  of graphs with maximum degree bounded. In order to prove Theorem 3 and Theorem 4, we shall establish the following Lemma, in which  $K_p(k)$  is the complete  $p$ -partite graph with  $k$  vertices in each part.

**Lemma 10** *For integers  $k \geq 1$  and  $p \geq 2$ , let  $t_p(k)$  be the maximum number of edges in a subgraph of  $K_p(k)$  that contains no  $K_p$ . Then*

$$t_p(k) = \left[ \binom{p}{2} - 1 \right] k^2.$$

**Proof.** By deleting all edges between a pair of parts of  $K_p(k)$ , we have the lower bound for  $t_p(k)$  as required. On the other hand, we shall prove by induction of  $k$  that if a subgraph  $G = G(V^{(1)}, \dots, V^{(p)})$  of  $K_p(k)$  contains no  $K_p$ , then  $e(G) \leq \left[\binom{p}{2} - 1\right] k^2$ . Suppose  $k \geq 2$  and  $p \geq 3$  as it is trivial for  $k = 1$  or  $p = 2$ . Furthermore, suppose that  $G$  has the maximum possible number of edges subject to this condition. Then  $G$  must contain  $K_p - e$  as a subgraph, otherwise we could add an edge and the resulting graph would still not contain  $K_p$ . Pick a vertex set  $X$  consisting of a vertex from each  $V^{(i)}$  for  $i = 1, 2, \dots, p$  such that  $e(X)$  is maximum among all such vertex subsets, and so  $e(X) = \binom{p}{2} - 1 = \frac{(p+1)(p-2)}{2}$ . We may suppose that  $X$  induces a complete graph of order  $p$  with an edge  $v_1 v_2$  missing, where  $v_1 \in V_1$  and  $v_2 \in V_2$ . Let  $Y = V(G) \setminus X$ , clearly each part of  $Y$  has  $k - 1$  vertices. Now, by noticing the fact that no vertex in  $V^{(i)} \cap V(Y)$  is adjacent to all the vertices of  $X \setminus \{v_i\}$  for  $i = 1, 2$  since  $G$  contains no  $K_p$ , we can safely deduce the desired upper bound of  $t_p(k)$  by a simple calculation, which completes the induction hypothesis hence the proof.  $\square$

**Lemma 11** *Let  $(A, B)$  be an  $\epsilon$ -regular pair of density  $d \in (0, 1]$ , and  $Y \subseteq B$  with  $|Y| \geq \epsilon|B|$ . Then there exists a subset  $A' \subseteq A$  with  $|A'| \geq (1 - \epsilon)|A|$ , each vertex in  $A'$  is adjacent to at least  $(d - \epsilon)|Y|$  vertices in  $Y$ .*

*Proof.* Let  $X$  be the set of vertices with fewer than  $(d - \epsilon)|Y|$  neighbors in  $Y$ . Then  $e(X, Y) < (d - \epsilon)|X||Y|$ , so  $d(X, Y) < d - \epsilon$ . Since  $(A, B)$  is  $\epsilon$ -regular, this implies that  $|X| < \epsilon|A|$ .  $\square$

**Proof of Theorem 3.** We will consider a red/blue edge coloring of  $K_p(cn)$ . Denote by  $H_R$  and  $H_B$  the subgraphs spanned by red edges and blue edges, respectively. Note that a partition obtained by applying Theorem 2 for  $H_R$  is such a partition for  $H_B$ .

Let  $p = R(K_\Delta)$  as defined. Clearly, we can only consider graphs  $G = G_n$  in  $\mathcal{G}_\Delta$  with  $n \geq \Delta + 2$ . Choose  $\epsilon = \min\{\frac{1}{p^2}, \frac{1}{m}\}$ , where  $m$  is a positive integer such that

$$(1 - \Delta\epsilon)(1/2 - \epsilon)^\Delta m \geq 1 \quad \text{hence} \quad (1 - \Delta\epsilon)(1/2 - \epsilon)^\Delta \geq \epsilon.$$

Let  $M = M(\epsilon, m, p) > 2m$  be the integer determined by  $\epsilon$  and  $p$  in Theorem 2 for  $H_R$ . Finally, let  $c = mM$  which is a constant determined completely by  $\Delta$ . We shall show that either  $H_R$  contains  $G$  or  $H_B$  contains  $G$ , hence  $f(G; p) \leq cpn$ .

Let the vertex set of the  $K_p(cn)$  be  $V = V^{(1)} \cup \dots \cup V^{(p)}$  with  $|V_\ell| = cn$  for  $1 \leq \ell \leq p$ . There is a partition of  $V$ , in which each  $V^{(\ell)}$  is partitioned into  $\{V_1^{(\ell)}, \dots, V_k^{(\ell)}\}$  with  $||V_i^{(\ell)}| - |V_j^{(\ell)}|| \leq 1$  and  $m \leq k \leq M$ , and all but at most  $\epsilon k^2 \binom{p}{2}$  pairs  $(V_i^{(s)}, V_j^{(t)})$ ,  $1 \leq i, j \leq k$ ,  $1 \leq s \neq t \leq p$ , are  $\epsilon$ -regular.

Let  $F$  be the subgraph of  $K_p(k)$ , whose vertices are  $\{V_i^{(\ell)} \mid 1 \leq \ell \leq p, 1 \leq i \leq k\}$  in which a pair  $(V_i^{(s)}, V_j^{(t)})$  for  $s \neq t$  is adjacent if and only if the pair is  $\epsilon$ -regular in  $H_R$ . Then the number of edges of  $F$  is at least

$$(1 - \epsilon)k^2 \binom{p}{2} > \left[\binom{p}{2} - 1\right] k^2 = t_p(k).$$

By Lemma 10,  $F$  contains a complete graph  $K_p$ . Without loss of generality, assume that  $V_1^{(1)}, \dots, V_1^{(p)}$  are pairwise  $\epsilon$ -regular. Color an edge between a pair  $(V_1^{(s)}, V_1^{(t)})$  green if  $d(V_1^{(s)}, V_1^{(t)}) \geq 1/2$ , or white if  $d(V_1^{(s)}, V_1^{(t)}) < 1/2$ . As  $p = R(K_\Delta)$ , we have  $\Delta$  sets in  $\{V_1^{(1)}, V_1^{(2)}, \dots, V_1^{(p)}\}$  such that they form a monochromatic  $K_\Delta$ . We may assume that the color is green since otherwise we consider the graph  $H_B$ .

Relabeling the sets in the partition if necessary, we assume that  $V_1^{(1)}, V_1^{(2)}, \dots, V_1^{(\Delta)}$  are pairwise  $\epsilon$ -regular in  $H_R$ , and  $d(V_1^{(s)}, V_1^{(t)}) \geq 1/2$ . Write

$$C_1 = V_1^{(1)}, C_2 = V_1^{(2)}, \dots, C_\Delta = V_1^{(\Delta)}.$$



Note that if  $Y_i \subseteq C_i$  with  $|Y_i| \geq (1 - \Delta\epsilon)(1/2 - \epsilon)^\Delta |C_i|$ , then  $|Y_i| \geq \epsilon |C_i|$ , which is the preparation for using Lemma 11, and

$$|Y_i| \geq (1 - \Delta\epsilon)(1/2 - \epsilon)^\Delta \frac{cn}{M} \geq n,$$

which will give us enough room to maneuver for constructing a color class of  $G$ .

Note that if a graph is neither a complete graph nor an odd cycle, then its chromatic number is at most  $\Delta(G)$ . For considered graph  $G = G_n$ , as  $n \geq \Delta + 2$  and  $\Delta \geq 3$ , we have  $\chi(G) \leq \Delta$ .

Assume that  $V(G) = \{u_1, u_2, \dots, u_n\}$ . We shall show that the red graph  $H_R$  contains  $G$  as a subgraph. We will choose  $v_1, v_2, \dots, v_n$  from the sets  $C_1, \dots, C_\Delta$ . Since  $\chi(G) \leq \Delta$ , so  $V(G)$  can be partitioned into  $\Delta$  color classes, which defines a map  $\phi: \{1, \dots, n\} \rightarrow \{1, \dots, \Delta\}$ , where  $\phi(i)$  is the color of vertex  $u_i$ . Our aim is to define an embedding  $u_i \rightarrow v_i \in C_{\phi(i)}$ , such that  $v_i v_j$  is an edge of  $H_R$  whenever  $u_i u_j$  is an edge of  $G$ .

Our plan is to choose the vertices  $v_1, \dots, v_n$  inductively. Throughout the induction, we shall have a target set  $Y_i \subseteq C_{\phi(i)}$  assigned to each  $i$ . Initially,  $Y_i$  is the entire  $C_{\phi(i)}$ . As the embedding proceeds,  $Y_i$  will get smaller and smaller. Some vertices will be deleted in procedure. But any  $C_{\phi(i)}$  will really have some vertices deleted at most  $\Delta$  times. To make this approach work, we have to ensure  $Y_i$  do not get too small.

Let us begin the initial step. Set

$$Y_1^0 = C_{\phi(1)}, Y_2^0 = C_{\phi(2)}, \dots, Y_n^0 = C_{\phi(n)}.$$

Note that  $Y_i^0$  and  $Y_j^0$  are not necessarily distinct sets.

We then begin the first step by considering  $u_1$ , for which  $v_1$  will be selected from  $Y_1^0$ , and its neighbors,  $u_\alpha, \dots, u_\beta$ , say. Suppose that the degree of  $u_1$  is  $d$ . By using Lemma 11 repeatedly, we know that there exists a subset  $Y_1^1 \subseteq Y_1^0$  with  $|Y_1^1| \geq (1 - d\epsilon)|Y_1^0| \geq n$ , such that each vertex in  $Y_1^1$  has at least  $(1/2 - \epsilon)|Y_j^0|$  neighbors in  $Y_j^0$ , where  $j = \alpha, \dots, \beta$ . Choose an arbitrary vertex  $v_1$  from  $Y_1^1$ . For  $j = \alpha, \dots, \beta$ , define  $Y_j^1$  be the neighborhood of  $v_1$  in  $Y_j^0$ . For  $j \geq 2, j \neq \alpha, \dots, \beta$ , define  $Y_j^1 = Y_j^0$ , that is, no vertices are deleted from such  $Y_j^0$ . In this step,  $v_1$  has been chosen and it completely adjacent to  $Y_j^1$  in  $H$  whenever  $u_1$  and  $u_j$  are adjacent in  $G$ .

In a general step, we consider  $u_i$  and its neighbors. We will choose  $v_i$  for  $u_i$  from  $Y_i^{i-1}$ . Suppose that  $u_i$  has  $d_1$  neighbors in  $\{u_1, \dots, u_{i-1}\}$ , and  $d_2$  neighbors,  $u_\alpha, \dots, u_\beta$ , say, in  $\{u_{i+1}, \dots, u_n\}$ . Then  $d_1 + d_2 \leq \Delta$ , and  $|Y_i^{i-1}| \geq (1/2 - \epsilon)^{d_1} |Y_i^0|$ . That is to say, the current set  $Y_i^{i-1}$  are obtained from  $Y_i^0$  by deleting some vertices  $d_1$  times before this step. By using Lemma 11 repeatedly again, we have a subset  $Y_i^i \subseteq Y_i^{i-1}$  with  $|Y_i^i| \geq (1 - d_2\epsilon)|Y_i^{i-1}|$  so that each vertex in  $Y_i^i$  has at least  $(1/2 - \epsilon)|Y_j^{i-1}|$  neighbors in  $Y_j^{i-1}$ , where  $j = \alpha, \dots, \beta$ . Since

$$\begin{aligned} |Y_i^i| &\geq (1 - d_2\epsilon)|Y_i^{i-1}| \geq (1 - d_2\epsilon)(1/2 - \epsilon)^{d_1} |Y_i^0| \\ &\geq (1 - \Delta\epsilon)(1/2 - \epsilon)^\Delta |C_i| \geq n, \end{aligned}$$

we can choose a vertex  $v_i$  from  $Y_i^i$ , which is distinct from  $v_1, \dots, v_{i-1}$  that have been chosen before this step, and some may be from  $Y_i^i$ . For  $j = \alpha, \dots, \beta$ , define  $Y_j^i$  to be the neighborhood of  $v_i$  in  $Y_j^{i-1}$ . For  $j \geq i+1, j \neq \alpha, \dots, \beta$ , define  $Y_j^i = Y_j^{i-1}$ , that is, no vertices are deleted from such  $Y_j^{i-1}$ . Note that  $v_i$  is adjacent to any  $v_j$ , where  $j < i$  and  $u_j$  is adjacent to  $u_i$ , and  $v_i$  is completely connected with each set  $Y_j^i$ , in which a neighbor of  $v_i$  will be selected after this step.

It is easy to check that the condition for using Lemma 11 can be satisfied since  $(1 - \Delta\epsilon)(1/2 - \epsilon)^\Delta \geq \epsilon$ . We thus finished the general step hence the proof of Theorem 3.  $\square$

**Proof of Theorem 4.** For  $p = R_r(K_\Delta)$ , take  $\epsilon = \min\{\frac{1}{p^2}, \frac{1}{m}\}$ , where  $m$  is an integer such that

$$(1 - \Delta\epsilon)(1/r - \epsilon)^\Delta m \geq 1.$$

In the proof, we use Lemma 6. We shall have  $p$  sets, say  $V_1^{(1)}, \dots, V_1^{(p)}$ , such that every pair  $(V_1^{(s)}, V_1^{(t)})$ ,  $1 \leq s < t \leq p$ , is  $\epsilon$ -regular in each monochromatic graph. Connecting this pair with color  $\ell$  if its edge density is at least  $1/r$  in the monochromatic graph in color  $\ell$ ,  $1 \leq \ell \leq r$ . Then we have a  $r$ -edge coloring of  $K_p$ , which implies a monochromatic  $K_\Delta$  in some color, say the color  $a$ . Hence we obtain  $\Delta$  sets, say  $V_1^{(1)}, \dots, V_1^{(\Delta)}$ , such that each pair  $(V_1^{(s)}, V_1^{(t)})$ ,  $1 \leq s < t \leq \Delta$ , is  $\epsilon$ -regular in monochromatic graph of color  $a$ , and the edge density of the pair is at least  $1/r$  in this color. The remaining proof is similar to that for Theorem 3.  $\square$

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